Int. J. Solids Structures, 1973, Vol. 9, pp. 495 to 506. Pergamon Press. Printed in Great Britain

ANALYSIS OF MULTI-BEAM SLABS

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Abstract—A simple difference equation model of a multi-beam slab is derived. A closed form solution is presented for each of two fundamental boundary conditions in the form of mixed finite and infinite trigonometric series. A homogeneous solution is derived and used to solve the problem of a slab supported by flexible edge beams. The solutions are numerically illustrated.

NOTATION

A_{ik}	coefficient in homogeneous solution
C_k	parameter in finite series
e	half-width of beam
E	Young's modulus
G	shear modulus
i	infinite summation index
I	moment of inertia of beam cross-section
I_ω	warping moment of inertia
k	finite summation index
K_t	torsional stiffness factor
$K^w(z, s)$, $K^\theta(z, s)$	beam kernel functions
K_i^w, K_i^θ, $\bar{K}_i^w$	kernel function coefficients
L	span length
m	number of beams
$M(r, z)$	applied torque
$M_i(r)$, M_{ik}	series coefficients of applied torque
$P(r, z)$	applied load
$P_i(r)$, P_{ik}	series coefficients of applied loads
r	discrete variable labelling joints and beams
s	dummy integration variable
S_k	parameter in finite series
V_i^s, $V_i^{a/s}$, V_{ik}^b	series coefficients of boundary shear
$V(r, z)$	interactive shear
$V_i(r)$, V_{ik}, V_{ik}^h	series coefficients of interactive shear
$W(r, z)$	deflection
$W_i(r)$, W_{ik}, W_{ik}^h	series coefficients of deflection
z	co-ordinate along beam axis
α_i	factor in arguments of series functions
γ_i	bending-to-torsion ratio parameter
$\delta(z, s)$	Dirac delta
$\theta(r, z)$	rotation
$\theta_i(r)$, θ_{ik}, θ_{ik}^h	series coefficients of rotation
π	circular ratio, 3.14159
σ^s, $\sigma^{a/s}$	summation of loading effects
$\phi(r)$	weighting function
Δ_r	forward difference operator
∇_r	backward difference operator
$\boxbslash_r$	second central difference operator

1. INTRODUCTION

A MULTI-BEAM slab consists of a number of prefabricated beams placed side by side and connected by longitudinal shear connectors as shown in Fig. 1. Such a system is attractive to the engineer requiring a flat, two-dimensional structure. It combines the ease of fabrication of the one-dimensional element with the bi-directional efficiency of the plate.

Past research into the behavior of multi-beam slabs was concerned mostly with their application as bridge decks. A recent report by Sanders and Elleby [1], devoted in large part to a state-of-the-art study of current methods of analysis, contains a comprehensive bibliography and is recommended to those desiring to make a thorough review of earlier efforts in the area.

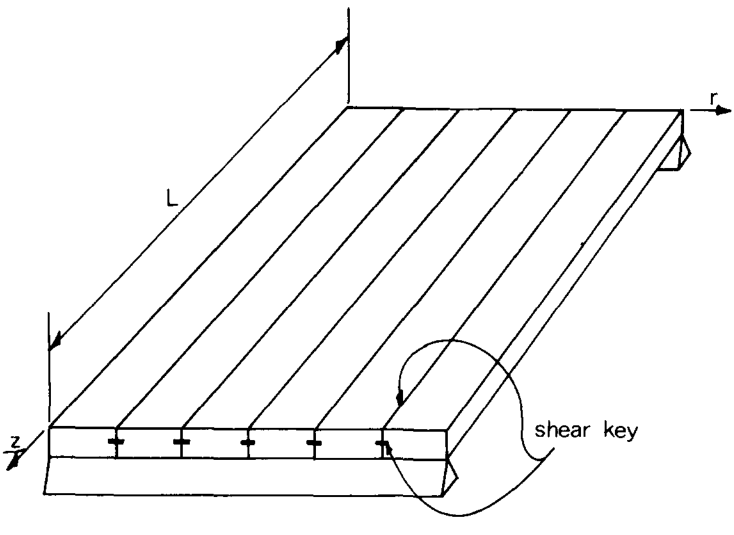

FIG. 1. Multi-beam slab.

Current methods may generally be placed into one of three categories. Some writers [2] propose that the system be analyzed by solving simultaneous equations for the unknown coefficients of the harmonic components of the forces interacting between the beams. This process must be repeated for each harmonic term until satisfactory convergence is obtained. This may be time-consuming, especially for irregular loading patterns which generally produce somewhat slowly convergent results. A second approach, which is analogous to the orthotropic plate approximation for ribbed plates, may be called the equivalent continuum method [3]. In this approach, one solves the differential equation of a plate which lacks moment resistance in one direction. An objection to this method may be made on the basis that the continuous model does not truly represent the discrete nature of the structure. A third method is the use of simplified empirical formulas based on experience or compilations of theoretical results. One such formula is found in the AASHO specifications for highway bridges [4].

The method developed in this paper, involving the solution of difference equations, combines the advantages of the first two methods mentioned above. That is, it preserves the discrete nature of the problem while admitting a closed form solution. (A somewhat similar method has been previously used by Dean and Omidvaran [8] to analyze ribbed plates.)

2. DERIVATION OF MATHEMATICAL MODEL

In the following development it is assumed that either the beams are proportioned or the connections are detailed such that in-plane effects are negligible. That is, the only component of force interacting between the beams is transverse shear.

Consider a multi-beam slab made up of m identical prismatic beams, the cross section of each symmetric about its vertical centerline. Denote the beams and joints by the discrete variable r as shown in Fig. 2. Let the positive directions for the forces and displacements of the r-th beam be as shown in Fig. 3. The vertical deflection and torsional rotation are given by

$$W(r, z) = \int_0^L [P(r, s) + \nabla_r V(r, s)] K^w(z, s) \, ds \tag{1}$$

and

$$\theta(r, z) = \int_0^L [M(r, s) - e(\nabla_r - 2) V(r, s)] K^\theta(z, s) \, ds \tag{2}$$

in which $K^w(z, s)$ is the deflection at z due to a unit impulse load at s and $K^\theta(z, s)$ is the torsional rotation at z due to a unit impulse torque at s. Also, ∇_r is the first backward difference operator, i.e.,

$$\nabla_r V(r, s) = V(r, s) - V(r-1, s). \tag{3}$$

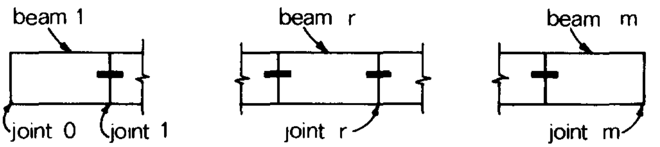

FIG. 2. System of labeling beams and joints.

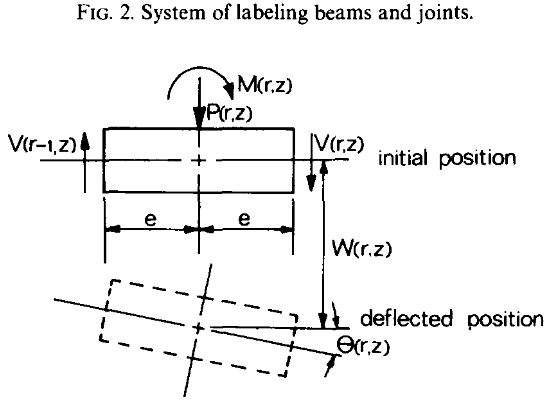

FIG. 3. Positive directions for forces and displacements.

The requirement that the connected points of two adjacent beams have the same vertical displacement may be expressed by

$$\Delta_r W(r, z) - e(\Delta_r + 2)\theta(r, z) = 0 \tag{4}$$

where Δ_r is the first forward difference operator; i.e.,

$$\Delta_r W(r, z) = W(r+1, z) - W(r, z). \tag{5}$$

If force-deformation relations (1) and (2) are substituted into compatability equation (4) the result is

$$\int_0^L \{[\Delta_r P(r, s) + \boxslash_r V(r, s)]K^w(z, s) - e[(\Delta_r + 2)M(r, s) + e(\boxslash_r + 4)V(r, s)]K^\theta(z, s)\}\, ds = 0 \tag{6}$$

where $\boxslash_r$ is the second central difference operator; i.e.,

$$\boxslash_r V(r, s) = V(r-1, s) - 2V(r, s) + V(r+1, s). \tag{7}$$

The beam kernel functions, derived in Appendix 1, are given by

$$K^w(z, s) = \sum_{i=1}^{\infty} K_i^w \sin(\alpha_i z)\sin(\alpha_i s) \tag{8}$$

$$K^\theta(z, s) = \sum_{i=1}^{\infty} K_i^\theta \sin(\alpha_i z)\sin(\alpha_i s) \tag{9}$$

where

$$\alpha_i = i\pi/L. \tag{10}$$

It is necessary to expand the various force and displacement functions into infinite series as shown below.

$$W(r, z) = \sum_{i=1}^{\infty} W_i(r)\sin(\alpha_i z) \tag{11}$$

$$\theta(r, z) = \sum_{i=1}^{\infty} \theta_i(r)\sin(\alpha_i z) \tag{12}$$

$$P(r, z) = \sum_{i=1}^{\infty} P_i(r)\sin(\alpha_i z) \tag{13}$$

$$M(r, z) = \sum_{i=1}^{\infty} M_i(r)\sin(\alpha_i z) \tag{14}$$

$$V(r, z) = \sum_{i=1}^{\infty} V_i(r)\sin(\alpha_i z). \tag{15}$$

If the above five series are substituted into equation (6), the result (after carrying out the indicated integration, matching coefficients of $\sin(\alpha_i z)$, and simplification) is the following difference equation involving the Euler coefficients of the infinite series for the various functions.

$$[(1-\gamma_i)\boxslash_r - 4\gamma_i]V_i(r) = (\gamma_i/e)(\Delta_r + 2)M_i(r) - \Delta_r P_i(r). \tag{16}$$

The parameter γ_i is given by

$$\gamma_i = e^2 K_i^{\theta}/K_i^{w}. \tag{17}$$

Substitution of the series into equations (1) and (2) now gives the force-deformation relations, also in terms of the coefficients of the infinite series, i.e.,

$$W_i(r) = (LK_i^w/2)[P_i(r) + \nabla_r V_i(r)], \tag{18}$$

and

$$\theta_i(r) = (LK_i^{\theta}/2)[M_i(r) - e(\nabla_r - 2)V_i(r)]. \tag{19}$$

Equation (16) is a linear second-order difference equation with constant coefficients, and, together with the relevant boundary conditions, governs the behavior of a multi-beam slab under the assumptions of this paper.

3. SOLUTION FOR TWO BASIC BOUNDARY CONDITIONS

Equation (16) may be solved for two important boundary conditions. The case when the slab is simply supported along the joint lines $r = 0$ and $r = m$ is handled as follows. Assume the following finite series expressions for the coefficients of the infinite series:

$$P_i(r) = \sum_{k=1}^{m} P_{ik} \sin \frac{k\pi(r-\frac{1}{2})}{m} \tag{20}$$

$$M_i(r) = \sum_{k=0}^{m-1} M_{ik} \cos \frac{k\pi(r-\frac{1}{2})}{m} \tag{21}$$

$$V_i(r) = \sum_{k=0}^{m} V_{ik} \cos \frac{k\pi r}{m}. \tag{22}$$

When these expressions are inserted into the governing equation and coefficients of $\cos(k\pi r/m)$ are matched, the resulting expression, after solving for V_{ik}, is

$$V_{ik} = \frac{S_k P_{ik} - (\gamma_i/e) C_k M_{ik}}{2(\gamma_i C_k^2 + S_k^2)}, \tag{23}$$

where

$$S_k = \sin \frac{k\pi}{2m} \tag{24}$$

$$C_k = \cos \frac{k\pi}{2m}. \tag{25}$$

Substitution of these expressions into the force-deformation equations gives the following expressions for the coefficients of the infinite displacement series. (Note that these expressions satisfy the simple support boundary conditions; i.e. the deflections are antisymmetric about the boundary joint lines $r = 0$ and $r = m$.)

$$W_i(r) = \sum_{k=1}^{m} W_{ik} \sin \frac{k\pi(r-\frac{1}{2})}{m} \tag{26}$$

$$\theta_i(r) = \sum_{k=0}^{m-1} \theta_{ik} \cos \frac{k\pi(r-\frac{1}{2})}{m} \tag{27}$$

where

$$W_{ik} = \frac{LK_i^w \gamma_i C_k}{2e}\left(\frac{eC_k P_{ik} + S_k M_{ik}}{\gamma_i C_k^2 + S_k^2}\right) \tag{28}$$

$$\theta_{ik} = \frac{\gamma K_i^\theta S_k}{2}\left(\frac{eC_k P_{ik} + S_k M_{ik}}{\gamma_i C_k^2 + S_k^2}\right). \tag{29}$$

The case where the slab is free along the joint lines $r = 0$ and $r = m$ is handled in a like manner. Assume these finite series expressions.

$$P_i(r) = \sum_{k=0}^{m-1} P_{ik} \cos\frac{k\pi(r-\frac{1}{2})}{m} \tag{30}$$

$$M_i(r) = \sum_{k=1}^{m} M_{ik} \sin\frac{k\pi(r-\frac{1}{2})}{m} \tag{31}$$

$$V_i(r) = \sum_{k=1}^{m-1} V_{ik} \sin\frac{k\pi r}{m}. \tag{32}$$

When these series are inserted into the governing equation and coefficients of $\sin(k\pi r/m)$ are matched, the resulting expression, after solving for V_{ik}, is

$$V_{ik} = -\frac{S_k P_{ik} + (\gamma_i/e) C_k M_{ik}}{2(\gamma_i C_k^2 + S_k^2)}. \tag{33}$$

Note that equation (32) satisfies the free edge boundary conditions $V_i(0) = V_i(m) = 0$. Substitution into equations (18) and (19) now gives

$$W_i(r) = \sum_{k=0}^{m-1} W_{ik} \cos\frac{k\pi(r-\frac{1}{2})}{m} \tag{34}$$

$$\theta_i(r) = \sum_{k=1}^{m} \theta_{ik} \sin\frac{k\pi(r-\frac{1}{2})}{m} \tag{35}$$

where

$$W_{ik} = \frac{LK_i^w \gamma_i C_k}{2e}\left(\frac{eC_k P_{ik} - S_k M_{ik}}{\gamma_i C_k^2 + S_k^2}\right) \tag{36}$$

$$\theta_{ik} = -\frac{LK_i^\theta S_k}{2}\left(\frac{eC_k P_{ik} - S_k M_{ik}}{\gamma_i C_k^2 + S_k^2}\right). \tag{37}$$

4. BOUNDARY SOLUTIONS

A homogeneous solution may be found in the following form:

$$V_i^h(r) = 1 + \sum_{k=1,3}^{m-1} A_{ik} \sin\frac{k\pi r}{m}. \tag{38}$$

The coefficient A_{ik} is found by substituting the above expression into the homogeneous governing equation. (It is necessary to expand unity into a finite series.) The result is

$$A_{ik} = \frac{-(2\gamma_i C_k/mS_k)}{\gamma_i C_k^2 + S_k^2}. \tag{39}$$

The homogeneous beam displacement coefficients $W_i^h(r)$ and $\theta_i^h(r)$ may be found by substitution into the force-deformation equations. This gives

$$W_i^h(r) = LK_i^w \sum_{k=1,3}^{m-1} A_{ik} S_k \cos\frac{k\pi(r-\frac{1}{2})}{m} \tag{40}$$

$$\theta_i^h(r) = eLK_i^\theta\left[1 + \sum_{k=1,3}^{m-1} A_{ik} C_k \sin\frac{k\pi(r-\frac{1}{2})}{m}\right]. \tag{41}$$

It should be noted that the above solution corresponds to the condition of physically antisymmetric unit harmonic boundary shears; that is, $V_i(0) = V_i(m) = 1$.

Similarly, homogeneous solutions corresponding to physically symmetric unit harmonic boundary shears are:

$$V_i^h(r) = 1 - \frac{2r}{m} + \sum_{k=2,4}^{m-1} A_{ik} \sin\frac{k\pi r}{m} \tag{42}$$

$$W_i^h(r) = LK_i^w\left[-\frac{1}{m} + \sum_{k=2,4}^{m-1} A_{ik} S_k \cos\frac{k\pi(r-\frac{1}{2})}{m}\right] \tag{43}$$

$$\theta_i^h(r) = eLK_i^\theta\left[\frac{(m-2r+1)}{m} + \sum_{k=2,4}^{m-1} A_{ik} C_k \sin\frac{k\pi(r-\frac{1}{2})}{m}\right]. \tag{44}$$

These homogeneous solutions may be used to solve the case where the edges of the slab are supported by an edge or stiffening beam. The symmetric and antisymmetric components of the boundary shear coefficients are given by

$$V_i^s = \tfrac{1}{2}[V_i(0) - V_i(m)] \tag{45}$$

$$V_i^{a/s} = \tfrac{1}{2}[V_i(0) + V_i(m)]. \tag{46}$$

These boundary shear coefficients may be found by equating the deflection of the slab boundary and that of the edge beam. The result is

$$V_i^{a/s} = \frac{(\gamma_i/e)\sum_{k=1,3}^{m}\left[\left(\frac{eC_kP_{ik} - S_kM_{ik}}{\gamma_i C_k^2 + S_k^2}\right)(C_k - S_k)\right]}{(\bar{K}_i^w/K_i^w) - 2[\sigma_i^{a/s} - \gamma_i(1+\sigma_i^{a/s})]} \tag{47}$$

$$V_i^s = \frac{(\gamma_i/e)\sum_{k=0,2}^{m}\left[\left(\frac{eC_kP_{ik} - S_kM_{ik}}{\gamma_i C_k^2 + S_k^2}\right)(C_k - S_k)\right]}{(\bar{K}_i^w/K_i^w) - 2[\sigma_i^s - \gamma_i(1+\sigma_i^s)]}, \tag{48}$$

in which

$$\sigma_i^{a/s} = \sum_{k=1,3}^{m-1} A_{ik} C_k S_k \tag{49}$$

$$\sigma_i^s = -\frac{1}{m} + \sum_{k=2,4}^{m-1} A_{ik} C_k S_k. \tag{50}$$

The edge beam flexural kernel function coefficient is denoted by $\bar{K}_i^w$. By adding the effects of the loads and the boundary shears the following expressions for the total effects may be written.

$$V_i(r) = \sum_{k=1}^{m-1} (V_{ik} + V_{ik}^b V_{ik}^h) \sin \frac{k\pi r}{m} \tag{51}$$

$$W_i(r) = \sum_{k=0}^{m-1} (W_{ik} + V_{ik}^b W_{ik}^h) \cos \frac{k\pi(r-\frac{1}{2})}{m} \tag{52}$$

$$\theta_i(r) = \sum_{k=1}^{m} (\theta_{ik} + V_{ik}^b \theta_{ik}^h) \sin \frac{k\pi(r-\frac{1}{2})}{m} \tag{53}$$

where V_{ik}, W_{ik} and θ_{ik} are given by equations (33), (36) and (37), respectively, and where

$$V_{ik}^b = \begin{cases} V_i^s & \text{if } k \text{ is even} \\ V_i^{a/s} & \text{if } k \text{ is odd} \end{cases} \tag{54}$$

$$V_{ik}^h = \frac{2C_k S_k(1-\gamma_i)}{m(\gamma_i C_k^2 + S_k^2)} \tag{55}$$

$$W_{ik}^h = -\frac{2LK_i^w \gamma_i \phi(r)}{m(\gamma_i C_k^2 + S_k^2)} \tag{56}$$

$$\theta_{ik}^h = \frac{2eLK_i^\theta S_k \phi(r)}{m(\gamma_i C_k^2 + S_k^2)}. \tag{57}$$

Note that equation (51) is not valid at the boundary joint lines $r = 0$ and $r = m$. The weighting function $\phi(r)$ is defined in Appendix 2.

5. NUMERICAL EXAMPLE

Consider a multi-beam slab of nine interior beams supported by two edge beams with the following properties: $m = 9$, $L = 480$ in., $e = 18$ in., $EI = 15 \times 10^6$ Kip/in^2, $GK_t = 20 \times 10^6$ Kip/in^2, $\bar{K}_i^w/K_i^w = 0.10$. Let the slab be loaded by a line load of intensity (0.006 in/Kip.) $\sin(\alpha_i z)$ applied along the center-line of the fifth beam. Equations (17), (A-6), and (A-11) give the following values: $\gamma_1 = 0.0104093$, $K_1^\theta = 4.86341 \times 10^{-6}$ rad/in/Kip., $K_1^w = 0.151378$ in/Kip. Expansion of the load into a finite series [note that $M_i(r) = 0$] and substitution into equations (33), (36) and (37) give the values shown in Table 1.

TABLE 1. INTERMEDIATE RESULTS

k	V_{1k} Kip/in.	W_{1k} in.	θ_{1k} rad
0	0.000000	0.242204	0.00000
2	0.018072	−0.035290	7.13587×10^{-4}
4	−0.102203	0.007057	-3.28985×10^{-4}
6	0.007671	−0.001675	1.61175×10^{-4}
8	−0.006767	0.000157	-0.49379×10^{-4}

From equation (48) the boundary shear coefficient is found to be $V_1^{\ddagger} = 0.016000$ Kip/in. The values of $V_1(r)$, $W_1(r)$ and $\theta_1(r)$, as found from equations (51) through (53) are shown in Table 2. Note that only about 8 per cent of the applied load is carried by the loaded beam.

TABLE 2. FINAL RESULTS

r	$V_1(r)$ Kip/in.	$W_1(r)$ in.	$\theta_1(r)$ rad
1	0.017677	0.060894	7.0756×10^{-4}
2	0.020097	0.087915	7.9362×10^{-4}
3	0.023362	0.118635	9.1307×10^{-4}
4	0.027611	0.154347	10.7090×10^{-4}
5	−0.027611	0.173624	0.0000

6. CONCLUSIONS

The equations developed herein have the advantage of giving a solution to multi-beam slab problems without recourse to time-consuming open-form methods or to continuum analogies. The mathematical model is simple, and the solutions are well suited for computation.

REFERENCES

[1] W. W. SANDERS and H. A. ELLEBY, *Distribution of wheel loads on highway bridges.* Natl. Coop. Highway Research Program Report 83, Highway Research Board (1970).
[2] J. E. DUBERG, N. KHACHATURIAN and R. E. FRADINGER, Method for analysis of multibeam bridges. *J. Struct. Div. Am. Soc. Civ. Engrs.* **86**, No. ST7, 109 (1960).
[3] A. ROESLI and R. E. WALTHER, The analysis of pre-stressed multi-beam bridges as orthotropic plates. *Proc. World Conf. on Prestressed Concrete*, San Francisco (1957).
[4] *Standard Specifications for Highway Bridges*, 10th edition. American Association of State Highway Officials (1969).
[5] T. V. GALAMBOS, *Structural Members and Frames*, p. 22 ff. Prentice-Hall (1968).
[6] D. L. DEAN and C. P. UGARTE, Field Solutions for two-dimensional frameworks. *Int. J. mech. Sci.* **10**, 315 (1968).
[7] F. B. HILDEBRAND, *Finite-Difference Equations and Simulations.* Prentice-Hall (1968).
[8] D. L. DEAN and C. OMIDVARAN, Analysis of ribbed plates. *J. Struct. Div. Am. Soc. Civ. Engrs.* **95** (ST3), 411 (1969).

APPENDIX 1. DERIVATION OF BEAM KERNEL FUNCTIONS

Consider a prismatic beam simply supported at each end for both torsion and flexure. The differential equation for flexure is

$$\frac{d^4 W(z)}{dz^4} = \frac{P(z)}{EI} \tag{A-1}$$

where $W(z)$ and $P(z)$ are the deflection and load, respectively. A solution satisfying the boundary conditions is

$$W(z) = \sum_{i=1}^{\infty} \frac{P_i}{EI\alpha_i^4} \sin(\alpha_i z) \tag{A-2}$$

where the P_i are defined by

$$P(z) = \sum_{i=1}^{\infty} P_i \sin(\alpha_i z). \tag{A-3}$$

The series for the Dirac delta function is well known and is given by

$$\delta(z, s) = \frac{2}{L} \sum_{i=1}^{\infty} \sin(\alpha_i s) \sin(\alpha_i z). \tag{A-4}$$

Thus the flexural kernel function (i.e., the solution for a unit impulse load) is given by

$$K^w(z, s) = \sum_{i=1}^{\infty} K_i^w \sin(\alpha_i s) \sin(\alpha_i z) \tag{A-5}$$

where

$$K_i^w = \frac{2}{LEI\alpha_i^4}. \tag{A-6}$$

The behavior of beams under the action of nonuniform torsion is somewhat more complicated than is the case for flexure. For an open thin-walled section, the differential equation is

$$EI_\omega \frac{d^4\theta(z)}{dz^4} - GK_t \frac{d^2\theta(z)}{dz^2} = M(z) \tag{A-7}$$

where $\theta(z)$ and $M(z)$ are the torsional rotation and moment, respectively. (The reader is referred to textbooks, for instance Ref. [5], for the derivation of the differential equations.) GK_t and EI_ω are the torsional stiffness and warping stiffness, respectively. A series solution satisfying the boundary conditions is

$$\theta(z) = \sum_{i=1}^{\infty} \frac{M_i}{\alpha_i^2(\alpha_i^2 EI_\omega + GK_t)}. \tag{A-8}$$

Again using the Dirac delta loading gives

$$K^\theta(z, s) = \sum_{i=1}^{\infty} K_i^\theta \sin(\alpha_i s) \sin(\alpha_i z) \tag{A-9}$$

where

$$K_i^\theta = \frac{2}{L\alpha_i^2(\alpha_i^2 EI_\omega + GK_t)}. \tag{A-10}$$

It should be noted that this solution must be considered approximate if the beam has other than an open thin-walled cross section. In the case of a closed or solid cross section, it is suggested that ordinary St. Venant torsion theory be used. That is, set EI_ω to zero to get

$$K_i^\theta = \frac{2}{LGK_t\alpha_i^2}. \tag{A-11}$$

APPENDIX 2. FINITE FOURIER SERIES AND DIFFERENCE OPERATIONS

The theory and application of infinite trigonometric series is well known. Less familiar is the subject of finite trigonometric series. The theoretical study of these series belongs to the calculus of finite differences and is beyond the scope of this paper. The interested reader is referred to the literature [6, 7]. A few pertinent results are given here as an aid to the reader.

Consider a function $F(r)$ of the discrete variable r. It may be shown that $F(r)$ can be represented in the interval $0 \leq r \leq m$ by the series

$$F(r) = \sum_{k=a}^{b} F_k T_k(r) \tag{A-12}$$

where $T_k(r)$ is one of certain suitable trigonometric functions. Given in Table A-1 are formulas for the Euler coefficients F_k for each $T_k(r)$ used in this paper. Also given are the limits of summations a and b.

TABLE A-1

$T_k(r)$	a	b	F_k
$\sin\dfrac{k\pi r}{m}$	1	$m-1$	$\dfrac{2}{m}\sum_{r=1}^{m-1} F(r)\sin\dfrac{k\pi r}{m}$
$\cos\dfrac{k\pi r}{m}$	0	m	$\dfrac{2\phi(k)}{m}\sum_{r=0}^{m} \phi(r)F(r)\cos\dfrac{k\pi r}{m}$
$\sin\dfrac{k\pi(r-\frac{1}{2})}{m}$	1	m	$\dfrac{2\phi(k)}{m}\sum_{r=1}^{m} F(r)\sin\dfrac{k\pi(r-\frac{1}{2})}{m}$
$\cos\dfrac{k\pi(r-\frac{1}{2})}{m}$	0	$m-1$	$\dfrac{2\phi(k)}{m}\sum_{r=1}^{m} F(r)\cos\dfrac{k\pi(r-\frac{1}{2})}{m}$

The weighting function $\phi(r)$ is defined as

$$\phi(r) = \begin{cases} 1 & \text{if } 0 < r < m \\ \frac{1}{2} & \text{if } r = 0 \text{ or } r = m. \end{cases} \tag{A-13}$$

It should be noted that in the case where $T_k(r)$ is equal to $\sin(k\pi r/m)$, then the finite series defines a function which is zero at the boundaries; i.e.,

$$F(0) = F(m) = 0. \tag{A-14}$$

It is convenient to have available a listing of the results of difference operations on these trigonometric functions. Listed below are the results of the difference operations used in this paper.

$$\nabla_r \sin\frac{k\pi r}{m} = 2S_k \cos\frac{k\pi(r-\frac{1}{2})}{m} \tag{A-15}$$

$$(\nabla_r - 2)\sin\frac{k\pi r}{m} = -2C_k \sin\frac{k\pi(r-\frac{1}{2})}{m} \tag{A-16}$$

$$\nabla_r \cos\frac{k\pi r}{m} = -2S_k \sin\frac{k\pi(r-\frac{1}{2})}{m} \tag{A-17}$$

$$(\nabla_r - 2)\cos\frac{k\pi r}{m} = -2C_k \cos\frac{k\pi(r-\frac{1}{2})}{m} \tag{A-18}$$

$$\Delta_r \sin\frac{k\pi(r-\frac{1}{2})}{m} = 2S_k \cos\frac{k\pi r}{m} \tag{A-19}$$

$$(\Delta_r + 2)\sin\frac{k\pi(r-\frac{1}{2})}{m} = 2C_k \sin\frac{k\pi r}{m} \tag{A-20}$$

$$\Delta_r \cos\frac{k\pi(r-\frac{1}{2})}{m} = -2S_k \sin\frac{k\pi r}{m} \tag{A-21}$$

$$(\Delta_r + 2)\cos\frac{k\pi(r-\frac{1}{2})}{m} = 2C_k \cos\frac{k\pi r}{m} \tag{A-22}$$

$$\boxbslash \cos\frac{k\pi r}{m} = -4S_k^2 \cos\frac{k\pi r}{m} \tag{A-23}$$

$$\boxbslash \sin\frac{k\pi r}{m} = -4S_k^2 \sin\frac{k\pi r}{m}. \tag{A-24}$$

(*Received* 28 *February* 1972; *revised* 22 *September* 1972)

Абстракт—Определяется несложная модель уровнения в конечных разностях для плоского перекрытия, состоящего из некоторого числа балок, связанных с собой и работающих на продольный сдвиг. Дается решение в замкнутом виде для каждого из двух основных граничных условий, в виде смешанных конечных и бесконечных тригонометрических рядов. Выводится однородное решение, которое используется для расчета задачи плоского перекрытия, опирающегося на упругих, краевых балках. Решения иллюстрированны графически.